Developing Mathematical Proficiency

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It has long been recognised that successful mathematical learning comprises much more than just knowledge of skills and procedures. For example, Skemp (1976) identified the advantages of teaching mathematics for what he referred to as "relational" rather than "instrumental" understanding. More recently, Kilpatrick, Swafford and Findell (2001) proposed five "intertwining strands" of mathematical proficiency, namely Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition. In Australia, the new Australian Curriculum: Mathematics (F–10), which will be implemented from 2013, has adapted and adopted the first four of these proficiency strands to emphasise the breadth of mathematical capabilities that students need to acquire through their study of the various content strands. This paper addresses the question of what types of classroom practice can provide opportunities for the development of these capabilities in elementary schools. It draws on data from a number of projects, as well as the literature, to provide illustrative examples. Finally, the paper argues that developing the full set of capabilities requires complex changes in teachers' pedagogy.

Introduction

> The purpose of mathematics … goes far beyond the establishment of mechanical skills. The ideas, principles, generalisations, and relationships which are taught, *as well as the skills,* are intended for purposes outside themselves and for use in situations quite unlike those in which they are learned. … In a word, we strive to teach understandings. (Brownell, 2007, p. 30)

It has long been recognised that successful mathematical learning comprises much more than just knowledge of skills and procedures. As early as 1944, in an article entitled *The progressive nature of learning mathematics*, the noted American mathematics educator William A. Brownell argued that teaching mathematics from a behaviourist perspective results, at best, in "pseudo-learning, memorisation, and superficial, empty verbalisation" (Brownell, 2007, p. 29). Brownell's argument was reprinted as one of two articles from the 1940s in the *100 Years of Mathematics Teacher* special issue of the *Mathematics Teacher* in 2007, where it is stated that "readers may be surprised at the modern ideas expressed by Brownell more than sixty years ago and will find numerous parallels to current discussions about conceptual understanding, problem solving, and the provision of quality mathematics education for all students" (p. 26).

Skemp (1976) identified the advantages of teaching mathematics for what he referred to as *relational* rather than *instrumental* understanding. Skemp argued that, while instrumental understanding may have short-term benefits, such as taking less time to learn skills and procedures, relational understanding is more adaptable to new tasks, is easier to understand, and is an appropriate goal in itself. He further contrasts instrumental and relational understanding, likening the former to providing learners with a series of steps to achieve certain goals, while the latter allows learners to build their own mental maps or schema, which, in principle, enable them to find their own paths to achieve a variety of mathematical goals.

More recently, Kilpatrick, Swafford and Findell (2001) proposed five "intertwining strands" of mathematical proficiency, namely:

- *Conceptual Understanding* – comprehension of mathematical concepts, operations and relations;

- *Procedural Fluency* – skill in carrying out procedures flexibly, accurately, efficiently and appropriately;

- *Strategic Competence* – ability to formulate, represent, and solve mathematical problems;

- *Adaptive Reasoning* – capacity for logical thought, reflection, explanation and justification; and

- *Productive Disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 116).

According to Kilpatrick et al. (2001), these strands are not independent but instead, "represent different aspects of a complex whole" (p. 116), thus leading to the notion of intertwined strands, as shown in Figure 1.
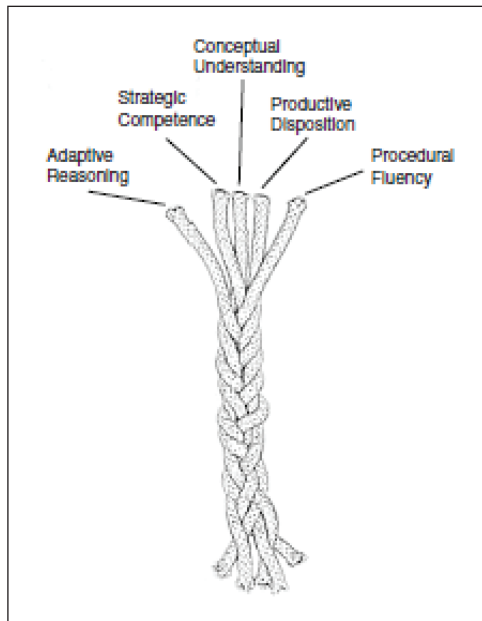

Figure 1. Intertwined strands of mathematical proficiency.
(Source: Kilpatrick et al., 2001, p.117)

In Australia, the new *Australian Curriculum: Mathematics (F–10)*, which will be implemented from 2013, has adapted and adopted the first four of Kilpatrick et al.'s (2001) proficiency strands to emphasise the breadth of mathematical capabilities that students need to acquire through their study of the various content strands. The four Australian proficiency strands are: *Understanding*, *fluency*, *problem solving*, and *reasoning* (Australian Curriculum Assessment and Reporting Authority, n.d.).

Similarly, the Singapore Mathematics Framework (Ministry of Education, Singapore, 2006, p.6) places problem solving at the centre of mathematics learning, with concepts, processes, metacognition, attitudes and skills placed around the sides of the pentagon (see Figure 2). This model, which is a minor modification of the original model published in 1990 (Dindyal, 2006, p. 181), first came to the author's attention in 1996 at the time the results of the *Third International Mathematics and Science Study* (TIMSS-95) became public. Kilpatrick (2011) described how he first became aware of the Singapore framework shortly after the publication of Kilpatrick et al. (2001) and notes the similarities, stating that "Both their framework and our strand model get at the same notion: that proficiency in mathematics is more than simply skill or understanding and that learners need to develop all five components simultaneously" (p.11).
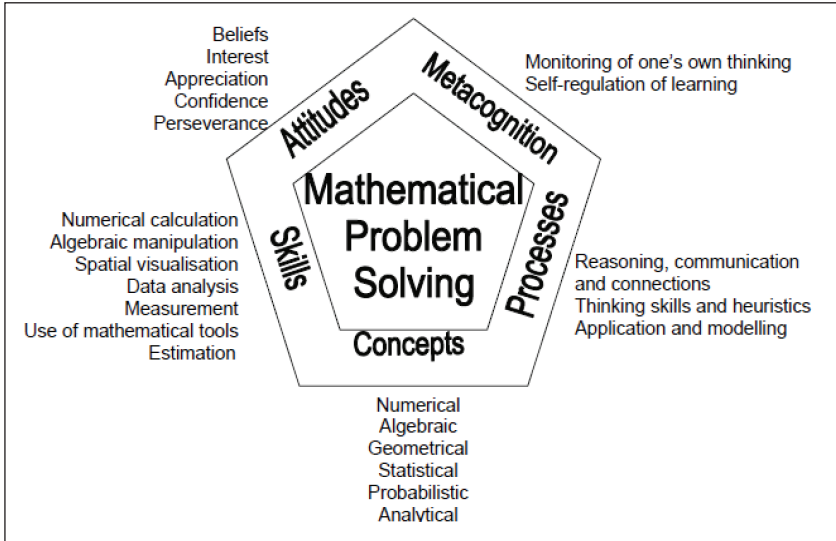
Figure 2. The Singapore Mathematics Framework.
(Source: Ministry of Education, Singapore, 2006, p.6)

While these capabilities have been adopted in many countries as evidence of mathematical proficiency, Ally (2011) questions the extent to which opportunities for developing these domains are present in teachers' pedagogies. In a study of four Grade 6 classes in South Africa, Ally looked for empirical evidence of the promotion of the five strands of mathematical proficiency. Findings revealed that "the extent to which the five strands of mathematical proficiency [were promoted was] … far below expectation" (p. 90). Over 90% of the 242 video-recorded five-minute lesson segments from 30 lessons contained opportunities for developing *procedural fluency*, with only 17% for *conceptual understanding*, 8% for *adaptive reasoning*, less than 2% for *strategic competence* and 20% for *productive disposition*. Further, the seemingly high frequency of opportunities for developing productive disposition was "mostly due to the inclusion of real world or out of class situations that teachers tried to link and incorporate into their lesson" (p. 90).

According to Schoenfeld (2007b), a growing body of literature shows that

> Students who experience skills-focused instruction tend to master the relevant skills, but do not do well on tests of problem solving and conceptual understanding [while] students who study more broad-based curricula tend to do reasonably well on tests of skills …[and] do much better … on assessments of conceptual understanding and problem solving. (p. 63)

Schoenfeld (2007b) further pointed out that "the nature of assessment (or testing) is critically important because *what you test is what you get* (WYTIWYG)" (p. 72). Unfortunately, however, high-stakes testing is constrained by many factors, including cost and test design that needs to conform to psychometric criteria such as reliability, construct validity, predictive validity, and test comparability, which greatly inhibits testing the full complement of strands of mathematical proficiency (Schoenfeld, 2007a).

There have been recent attempts to produce assessments that include a balance of mathematical tasks across the various strands of mathematical proficiency, which can then "provide an opportunity … to use these assessments as leverage to ensure that teachers nurture, observe, and monitor these important student behaviours" (Kepner & Huinker, 2012, p. 28). For example, in Malaysia, Khairani and Nordin (2011) developed and carried out a construct validation of a mathematics proficiency test for 14-year-old students assessing the three areas of conceptual understanding, procedural fluency, and strategic competence.

The purpose of this paper is to address the issue of the types of classroom practice that can provide opportunities for the development of Kilpatrick et al.'s (2001) strands of mathematical proficiency in elementary schools. It draws on data from a number of projects, as well as from literature, to provide illustrative examples.

Conceptual Understanding – The Circle Lesson

> *Conceptual understanding* refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful. They have organised their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know (Kilpatrick et al., 2001, p. 118).

There are many different meanings attributed to the term *conceptual understanding*. Adopting a framework "which characterizes conceptual understanding as attainment of an expert-like fluency with the conceptual structure of a domain", Richland, Stigler and Holyoak (2012, p. 190) note that the differing meanings attributed to the term *conceptual understanding* have "contributed to difficulty in changing teacher practices" (p. 190).

As Ally (2011) found, despite the continued rhetoric regarding the need for students to develop conceptual understanding, opportunities for this to happen do not occur frequently in regular classrooms. As a result, classroom discourse and the socio-mathematical norms associated with achieving quality dialogue have received considerable attention among mathematics educators (see, for example, Kazemi, 1998; Yackel & Cobb, 1996). As will be discussed in the next section, other researchers have also focused on the connections between conceptual understanding and procedural fluency. However, frameworks for effective teaching to support children's conceptual understanding also emphasise the need for tasks, which are mathematically challenging and significant (Fraivillig, 2001).

The following example, which is used here to illustrate a lesson based on a task designed specifically to focus on children's understanding of the concept of a circle, uses data from a research project, *Mathematics Classrooms Functioning as Communities of Inquiry: Models of Primary Practice* (see, for example, Groves & Doig, 2002). The lesson[1] was video-recorded in a Grade 3 class of eight children at the Japanese School of Melbourne. The school was chosen because of the Japanese practice of commencing each topic of work with a problem-solving lesson targeting a central concept for the topic.

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[1] This lesson is also described in Doig, Groves, and Fujii (2011).

The school teaches the Japanese curriculum in the Japanese language, to children whose parents are in Australia for periods of one or two years, with teachers usually coming to Australia for a period of three years. An interpreter was present throughout the lesson and assisted with the translation of the teacher interview after the lesson. According to the teacher, Mr. J, the main mathematics topic for the lesson was *the concept of a circle*.

The lesson began with Mr. J producing a pole for a game of quoits (a game where players attempt to throw rings onto a pole). Mr. J placed the pole in the centre of an open space in the classroom and asked three children (referred to as B1, G1 and B2) to stand at three marked places on a red line along one side of the room (see Figure 3). The children expressed concern that the game would be unfair, but mainly focus on the distance between children on the red line.

After discussing how to measure distances using a metre ruler to measure distances from the pole to the positions where B2, G1 and B1 were standing, Mr. J held up pre-cut coloured strips of paper, as shown in Figure 3, to show that the distances were different. He then defined the problem by asking, "How can we make the game fair?"
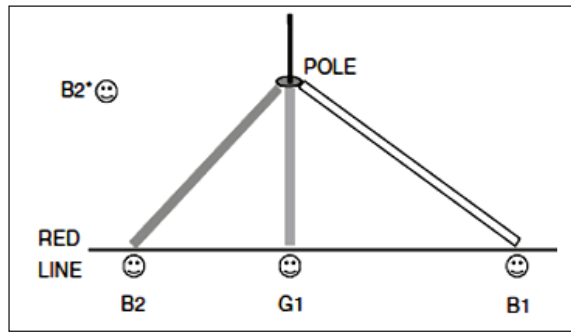

Figure 3. B2's solution for making the quoits game fair.

It was only after five more minutes, during which the children continued to try to find points to stand on the red line, that B2 suggested that two people can be at the same distance from the pole; he moved the yellow strip so that

one end was at the pole and the other at the point B2* on Figure 3. Mr. J then gave all the children a yellow strip and asked them to "think for themselves" to find where to stand so that everyone was at the same distance from the pole. The children were excited as no one would be at a disadvantage. This segment took 20 minutes in the 45-minute lesson.

Mr. J then represented the situation on a large sheet of paper on the board. He stuck a miniature pole on the paper and asked the children to use sticky yellow paper strips and dots to represent their positions (see Figure 4).
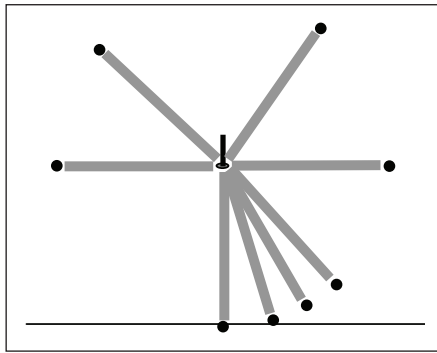

Figure 4. Paper representation of children's positions.

Mr. J: Look at the different positions. What do you notice?

G2: It's like a round circle [makes a circle shape with her hands].

G3: No, it's like a flower.

G1: If you follow the end of each yellow strip it will become a circle [traces a large circle on the desk with her finger].

Mr J: What if every student in the school took part? [adds more strips] …

B3: If there are many students standing round, maybe it's a circle.

Mr. J removed the pole and put another sheet of paper over the first with a circle drawn over where the dots were, and asked, "How many yellow points would we need? 100? 1,000,000?" He then put the word circle on the paper and elicited names for the centre, radius and diameter from the children.

The remaining 15 minutes of the lesson were taken up with the children working in pairs drawing circles. Initially many children chose to use a compass, even though Mr. J told them that they had not yet learnt how to use one and encouraged one girl, who said that she could use a yellow strip of paper or a plastic circle to trace round, to show him how. After about seven minutes, Mr. J asked the children to find a way to draw a circle without using a compass. A few minutes later Mr. J said, "Now everyone is tracing. Is there another way?" The children tried various ways, while Mr. J pivoted one of the yellow strips of paper around one end held by his finger. B2 excitedly cried out that he could do it and demonstrated drawing a circle by holding the middle of one end of his pencil case and tracing a circle with his finger in the hole at the other end. The children applauded and Mr. J demonstrated B2's method at the front of the class. The lesson closure was done with a few suggestions from the children on how to fix one end, culminating in the use of a drawing pin. Mr. J summed up by saying, "As you suggested, there are other ways of drawing a circle besides using a compass."

Mr. J highlighted the conceptual aspects, both in the lesson and in his responses in the subsequent interview. He stated that his aims were for the "children to have the *concept* of a circle and find *real* circular objects" [emphasis in original]. According to Mr. J, the most important aspect of the lesson in terms of children's learning was for the children to understand that the circle is a locus. This provides a sharp contrast to the Australian curriculum where at this level the focus is on the shape of a circle rather than on its underlying properties, with the emphasis being on procedural aspects such as recognising which shapes are circles and which are not based on observation.

The purpose for the children working in groups (in this case pairs) was "to facilitate discussions while working in groups", while the purpose of the whole class discussion was for children to "share ideas and strategies for solutions [demonstrating that] there are many different ways of thinking to reach the same conclusions". Mr. J further described his mathematics lessons as follows:

> Introductory lessons [to a topic] use materials. So this was typical. The introduction is very important and takes a lot of time. After that there is much practice, then we go on to calculations — a series of 3 or 4 lessons [per topic].

Mr. J concluded his interview with the comment that "Mathematics should be part of children's daily lives". According to Fisher, Hirsh-Pasek and Golinkoff (2012, p. 83), "in order for children to understand mathematics – and use it in meaningful ways – they must engage in personally meaningful activities that facilitate the learning process". In the 20-minute quoit activity, Mr J embedded the concept of a circle in a rich, intriguing, intrinsically motivating, problematic framework, by asking, "How can we make the game fair?"

Analysis of the focus group responses from the principals, teachers and mathematics educators who viewed the video of this lesson and another lesson from an "Australian" classroom, showed a high level of support for mathematics classrooms that functions as communities of inquiry, together with a realisation that current Australian practice falls far short of this goal. Principals and mathematics educators rated the cognitive demands of typical Australian lessons as low to very low and not challenging children, echoing Kazemi and Stipek's (2001) call for teachers to provide activities with high levels of *conceptual press* in order to stimulate children's conceptual understanding.

Procedural Fluency – Sporting Statistics

> Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy (Kilpatrick et al., 2001, p. 122).

Computational fluency means much more than speed and accuracy which have previously been regarded as its cornerstone. According to Kilpatrick et al. (2001) "*procedural fluency* refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121).

Furthermore, while accuracy and efficiency are often associated with pencil and paper computation, students need to develop their ability to perform mental computations flexibly and fluently. Such fluency needs to be based on number sense and the understanding of important concepts such as place value.

The following example[2], which is used here to illustrate ways in which students can be provided with opportunities to develop their written and mental procedural fluency, uses data from the *Calculators in Primary Mathematics* research project. In this example, the Grade 2 children had taken part in various sporting activities in the morning, which were used as the basis for a series of word problems for their mathematics lesson. The children were allowed to use any method they chose to find the answers to the problems, including pencil and paper, concrete materials, calculators and mental strategies. The lesson concluded with a lengthy sharing of solution strategies, part of which is reported below.

The video recording showed children firstly sharing their solutions to the following problem:

> Sarah's team scored 49 runs and Jason's team scored 63 runs.
> How many more runs did Jason's team score?

Two girls, Tennille and Penny, reported on their strategy.

Penny: Well, me and Tennille we were using our fingers and there weren't enough, so I used the calculator buttons and we ended up with 14.

While it is not obvious from the transcript, these girls counted on from 49 using their fingers, saying 50, 51, 52, … 59, and then, having no more fingers to use, they used the calculator buttons to record the extra four needed to count 60, 61, 62 and 63.

Jason, on the other hand, used a more sophisticated strategy:

Jason: Well, I worked out that 49 plus 11 was 60. Then I added on another 3 and that was 14.

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[2] For a video-recording of the lesson excerpts described here, see Groves and Cheeseman (1993).

As Jason explained his method, the teacher recorded it on the board in a manner not dissimilar to what is shown on an *empty number line* (see, for example, O'Loughlin, 2007, where it is referred to as the *open number line*). She commented:

Teacher: Right, so you said 49 plus 11. You knew that 49 plus 11 equalled 60 and then to get to 63 you added on another 3. Is that right?

Jason: Yes.

Teacher: And then what did you do?

Jason: Well, I found out that was 14.

Teacher: So you added those two together to get 14?

Jason: Yes.

The video recording showed children sharing their solutions to another problem:

> Everyone in the grade did 10 step-ups and 10 star-jumps.
> There are 26 children in the grade.
> How many step-ups and star-jumps?

The first pair to share their solutions counted around the room in tens. To find the total, they used their calculator to add 260 to 260.

Later, the teacher asked Jason to explain his solution.

Jason: Well I knew that 10, no, 20 tens is 200 and 6 tens is 60. So I knew that there will be 260 if I added them both together. Then I knew that they'd both be the same answer because they are the same sum.

Teacher: And had you worked out how many if you added them both together … what you'd get?

Jason: Well I didn't but I worked it out.

Teacher: Right. How did you work it out?

Jason: When I was on the floor I worked it out in my head.

Teacher: Can you explain – because you've got such an interesting way of working things out? Can you explain what you did?

Jason: Well I knew that 200 and 200 is 400. And I knew that 6 plus 6 is 12, and 12 tens is 120 so then it would be 520.

The last child to explain his solution explained that he did it "using my head. Because 26 times 10 … that makes 260".

This example illustrates the close inter-relationship between procedural fluency and conceptual understanding. For example, Jason not only demonstrated an extraordinary level of computational fluency, but also exhibited a deep understanding of place value, addition, and multiplication by 10, and was able to connect these in a way that enabled him to access the relevant ideas and use them to solve the problem in his own way.

Unlike written computations, where we normally rely on standard algorithms that are guaranteed to work for all examples, mental computation strategies are idiosyncratic, depending on the numbers being used and the person's ability to carry out the calculations. Choosing and using efficient mental strategies both requires understanding of our number system and helps support the development of efficient written strategies.

Sharing solution strategies as part of an extended, orchestrated discussion, where solutions are selected in order of increasing sophistication, is a characteristic of Japanese problem-solving lessons. This Australian teacher used a similar teaching strategy to help develop her students' computational fluency.

Strategic Competence – Snow White and the Seven Dwarfs

> *Strategic competence* refers to the ability to formulate mathematical problems, represent them and solve them. This strand is similar to what has been called problem solving and problem formulation in the literature of mathematics education and cognitive science, and mathematical problem solving, in particular, has been studied extensively. (Kilpatrick et al., 2001, p. 124)

Mathematical problem solving is central to mathematics learning. Genuine problem solving involves people in accepting the challenge of tackling an unfamiliar task for which they know no obvious solution. Of course, within mathematics, it also assumes that the problem is amenable to the application of some mathematics – something that also touches on the *productive disposition* strand of Kilpatrick et al.'s (2001) strands of mathematical proficiency.

For successful problem solving to take place, teachers need to create a classroom climate that supports problem solving. Problems need to be sufficiently challenging to interest students, but not so difficult that they get frustrated. When students become seriously stuck, teachers need to intervene in such a way that students retain ownership of the problem, and to avoid, in all but the most extreme cases, providing students with the solutions. Teachers should, whenever possible, listen to what their students have already discovered and attempt to build on their ideas.

The following example[3], which is used here to illustrate ways in which students can be provided with opportunities to develop their strategic competence, uses data from a research project *Talking Across Cultures: An International Collaborative Study of Student's Mathematical Explanations*. The lesson described below took place in Melbourne, Australia in the middle of the five- and six-year-old children's first year at school. Children sat on the floor while the teacher, Mrs. B, reminded the class that they had heard the story of *Snow White and the Seven Dwarfs* the previous day. She then put out a sheet of paper to represent a "long table" at which Snow White and the seven dwarfs sat for their dinner. She said that Snow White always sat at the head of the table, while the dwarfs sat at the two long sides, with a different number of dwarfs sitting on each side each day. Seven counters were used to represent the dwarfs. One child was asked to illustrate a possible way. She placed one counter on one side and the remaining six on the other side. The teacher then presented the problem on the board as shown below.

> How might the 7 dwarfs sit at the table?

> Did you find all the ways?
> How do you know?

The children were told they could paste coloured rectangles onto the paper to represent the table and draw "quick maths drawings"[4] – not ones with the dwarfs' "hair and hats and eyelashes" – as well as write numbers. Alternatively, they could use concrete materials and jotters, where they could

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[3] This lesson is also described in Doig, Groves, and Fujii (2011).
[4] Quotes are from the video recording of the lesson, with pseudonyms used for children's names.

record their answers because "your job is to find as many ways as possible".

The children worked individually or in pairs at their tables or on the floor for about 15 minutes, after which the teacher called the children back to the floor for a discussion of the different solutions.

Mrs. B commented that one child had said that he had found seven ways, while another had found six. She reminded them that they wanted to find all the possible ways. As individual children contributed different ways, she wrote their solutions on a piece of card, which she attached to a whiteboard as shown in Figure 5.

2 and 5
7 and 0
4 and 3
3 and 4
6 and 1

1 and 6
0 and 7
5 and 2

Figure 5. The children's different seating arrangements for the seven dwarfs.

The teacher commented that it was very difficult to see whether all the ways were represented and suggested that they look at an ordering that had been used by one of the girls, Melody (see Figure 6). When asked to try to find a pattern, the children replied that "it's just the opposite".

4 and 3
3 and 4
5 and 2
2 and 5

0 and 7
7 and 0
1 and 6
6 and 1

Figure 6. Melody's first ordering of the seating arrangements.

After some discussion with Melody as to whether she had actually moved the counters around (which she had done) or had just written the numbers "the other way around", the teacher noted that there was "another pattern we could make". When none of the children volunteered such a pattern, she showed the class the second pattern that Melody had made (see Figure 7).
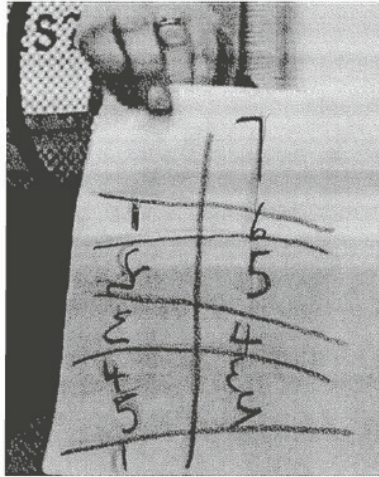

Figure 7. Melody's second ordering of the seating arrangements.

Mrs. B then reproduced Melody's ordering on the board by writing the first two arrangements on the board and asking different children to supply the remainder: "she's making a pattern; 0 and 7, 1 and 6 … Have a look what's happening: 0, 1, what do you think comes next?" When the children got to 3 and 4, the teacher asked Ivy, "What is the pattern on this [right] side?" and Ivy replied, "The pattern is 7, 6, 5, 4, like counting backwards … from 7". After asking another child for the pattern on the left hand side, the teacher and children completed the list of arrangements. Mrs. B asked the children whether there was anything different they could have done and, just to make sure, she told the children to go back to their tables and tell her if they had an arrangement that was missing from the list – but to make sure there were still seven dwarfs!

Mrs. B then told the children to come back to the floor and asked, "Have we found all the ways?" to which the children chorused, "Yes!" She continued, "Nobody else got any more at their tables for seven … but how do we *know* we've got all the ways?" One boy replied, "We've used all the numbers". The teacher confirmed this, discussed with the children why there could not be more, and asked them again how many ways there were for seven dwarfs. She then asked what would happen if "at the three pig's house" there were eight people – how many different ways could they sit? After Ivy answered, "Nine", the teacher asked what if there were ten visitors and Caitlin replied eleven ways. "So what if we had all 24 children in the class sitting at a very long table?" Ivy answered 25. The discussion continued:

Mrs. B:	What is the pattern? How did you know each time without doing it? When there were 7 people there were 8 ways. When there were 8 people there were 9 ways. When there were 10 people there were 11 ways. When there were 24 there were …25 ways. What's the pattern? … How did you know without doing it each time? … What if there 100 people?
Child:	101
Mrs. B:	What if there were 300 people?
Children:	400, 500, 104
Caitlin:	301

The children came to the board to try to write 301 – the fourth child wrote it correctly after the first three wrote 1Ɛ1 (131 with the "3" reversed), 3001, and 131, and the lesson ended with a discussion about which of these was correct. In total, the discussion of the children's solutions, how they knew they had found all the ways, and the discussion of the number of ways for different numbers of people, took approximately 25 minutes.

In a case study of one sixth-grade teacher's practices that supported the development of students' strategic competence, Özdemir and Pape (2012, p. 163) identified the following features: "*(a) allowing autonomy and shared responsibility …, (b) focusing on student understanding, (c) creating contexts for students … to exercise strategic behaviour,* and *(d) helping students to personalise strategies by recognising their ideas and strategic behaviours".* Mrs. B provided these very young students with the opportunity to exercise autonomy and strategic behaviour by letting them choose their own representations and strategies to solve the problem. Her extensive and clearly orchestrated sharing of student solutions, much like that encountered in Japanese structured problem-solving lessons, allowed students to gain ownership of their strategies. Arguably, these four features were present not only in this lesson, but also in Mrs. B's lesson, *The Fireman's Ladder,* which is described in the next section.

When asked about the goals for this lesson, Mrs. B focussed on developing problem solving processes, such as working systematically, as well as looking for patterns. In Japan, such a lesson might be termed a "jump-in lesson" to indicate that it could take place at many different points in the curriculum sequence.

Adaptive Reasoning – The Fireman's Ladder

> *Adaptive reasoning* refers to the capacity to think logically about the relationships among concepts and situations. Such reasoning is correct and valid, stems from careful consideration of alternatives, and includes knowledge of how to justify the conclusions. In mathematics, adaptive reasoning is the glue that holds everything together. (Kilpatrick et al., 2001, p.129)

According to the *Australian Curriculum: Mathematics (F–10)* (Australian Curriculum Assessment and Reporting Authority, n.d.), "Students are reasoning mathematically when they explain their thinking, when they

deduce and justify strategies used and conclusions reached, … when they prove that something is true or false and when they compare and contrast related ideas and explain their choices".

In the lesson[5] described below, five- and six-year-old children in their first year of school were given the opportunity to explain their thinking and justify their strategies. Mrs. B was the same teacher as for the *Snow White* lesson, but the *Fireman's Ladder* lesson took place several years earlier. After a discussion about *doubling* and also what is meant by *the middle*, the children were presented with the following problem:

The Fireman's Ladder
A fireman was standing on the middle rung of a ladder.
He goes up three more rungs to get to the top.
How many rungs are there altogether on the ladder?

The children were given a large sheet of paper with the problem written on it. They also had access to concrete materials, such as *Lego* figures to represent firemen, and straws and coloured sticks to represent the rungs of a ladder, as well as crayons and pencils. They were told that if they finished the problem they could try another one with five more rungs to get to the top, or another one with the number of rungs to the top being a number of their own choosing.

After working for some time at their tables, the children sat in a circle on the floor and different children were invited to share their solution strategies, with the children often illustrating their use of concrete materials on the floor.

As part of this discussion of solutions, the class discussed Megan's solution for the case of the fireman who sees ten more rungs to the top.

Mrs B: Now do you know what Megan did? She said she wasn't sure how many rungs on the ladder altogether. … She was about to make it with these sticks, weren't you? But then I said have a guess. And what did you say?

Megan:	21.
Mrs B:	What do you think about that guess? She had 10 more to get to the top. What do you think about that guess?
Megan:	[Says nothing]
Mrs B:	What do you think Jonathon?
Jonathon:	That it was, um, nearly correct.
Mrs B:	Nearly correct? OK I'll write it up here so we don't forget what you said. … OK she said 21. She had 10 more to get to the top on her ladder and her guess was 21. … OK so Jonathon thinks nearly. … Why do you think nearly Jonathon?
Jonathon:	Because I think it would be one less. …
Mrs B:	So what number do you think it would be?
Jonathon:	20.
Mrs B:	Why do you think it would be 20?
Jonathon:	Because 10 plus 10 is 20.
Mrs B:	What do you think Megan?
Megan:	Well because there's one in the middle and 10 plus 10 is 20 and there's 10 on each side.
Mrs B:	What's he forgotten in his way of doing it?
Megan:	There's one in the middle.

The teacher then suggested to Jonathon to use sticks. While he was putting out the sticks (which took several minutes) the teacher asked the class to put their hands up if they agreed with Megan that there were 21 rungs. Many children did so. She then asked those who agreed with Jonathon to put their hands up. A few did so.

After another child shared his solution, the focus moved back to Megan and Jonathon.

Mrs. B:	Now Jonathon, we had to check because you and Megan had different ideas, didn't you?
Jonathon:	Yeah.
Mrs B:	OK on the middle rung of the ladder. Now Jonathon you decided that in her story there were 20 rungs on the ladder altogether. Is that right?

Jonathon:	Mmm.
Mrs B:	And Megan thinks that there are 21. We have to sort out which one is right. OK. So let's talk about your one. Did you get 20 altogether here?
Jonathon:	No.
Mrs B:	You didn't? What did you find out?
Jonathon:	I found that there was 21.
Mrs B:	OK. Well how did you change your mind?
Jonathon:	Because I included the one in the middle.

Unlike the classes in Ally's (2011) study, where a mere 8% of lesson segments were found to provide opportunities for developing adaptive reasoning, Mrs B's lessons were characterised by being rich in such opportunities. In another segment of this lesson, she even asked these very young children "How could you prove it?" As in the work of Yackel and Cobb (1996), Mrs. B's lessons consisted typically of two types of activity: Teacher directed whole class activities, and small group activities. The whole class discussion emphasised children's explanations of their solution methods, with the expectation that children would be able to justify their solutions rather than just present them to the class as alternative methods.

Productive Disposition – The Bus Problem

> If students are to develop conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning abilities, they must believe that mathematics is understandable, not arbitrary; that, with diligent effort, it can be learned and used; and that they are capable of figuring it out. Developing a productive disposition requires frequent opportunities to make sense of mathematics, to recognise the benefits of perseverance, and to experience the rewards of sense making in mathematics. (Kilpatrick et al., 2001, p.131)

Many students see school mathematics as being totally divorced from their real life experiences and fail to see how mathematics can be a powerful tool in everyday life. Schoenfeld (2007b, p. 69) discussed the results of a nationwide United States survey of 45 000 students who were presented with the following version of the well-known bus problem:

> An army bus holds 36 soldiers.
> If 1128 soldiers are being bussed to their training site, how many buses are needed?

Schoenfeld reported that "29% gave the answer '31 remainder 12'; 18% gave the answer '31'; 23% gave the correct answer, '32'; and 30% did the computation incorrectly. A full 70% of the students did the computation correctly, but only 23% of the students rounded up correctly (p. 69). He attributed this behaviour to students believing that mathematics is meaningless, having nothing to do with the real world, and that they cannot be expected to understand it but only memorise it. Thus students' beliefs are a serious barrier to their developing productive dispositions.

Schoenfeld's solution to this significant problem is to advocate assessment that focuses on all aspects of mathematical proficiency, based on the *What You Test Is What You Get* (WYTIWYG) principle referred to earlier in this paper.

It is interesting to note that the only one of Kilpatrick et al.'s (2001) strands of mathematical proficiency missing from the new Australian Curriculum is *productive disposition*. Perhaps this is not surprising given Kilpatrick's (2011, p. 11) comment that in the *Mathematics Learning Study*, which led to Kilpatrick et al.'s (2001) report, some mathematicians resisted having productive disposition as one of the five strands, while teachers believed that it was critical for students to have a productive disposition towards mathematics if the other proficiency strands were to be met.

As Siegfried (2012, p.199) notes, "although *productive disposition* in isolation is a little-researched construct, connection of productive disposition to the other four strands of mathematical proficiency … is completely unresearched".

Conclusion

> Automata or thinkers? Which are we trying to develop? Society's demands are changing, and will continue to change, decade by decade—thus students need to develop flexibility and adaptability in using skills and concepts, and in self-propelled learning of new ones. (Burkhardt, 2007, p. 84)

In this paper I have argued that Kilpatrick et al.'s (2001) strands of mathematical proficiency attempt to encompass the capabilities that mathematics educators want to develop in our students. I have tried to address the question of the types of classroom practice that can provide opportunities for the development of these strands of mathematical proficiency in elementary schools by drawing on data from a number of research projects, as well as from literature, to provide illustrative examples of lessons where opportunities to develop these capabilities exist.

Developing the full gamut of these capabilities is by no means an easy task. There has been extensive research (see for example, Kazemi & Stipek, 2001; Stein, Engle, Smith & Hughes, 2008; Yackel & Cobb, 1996) into the complexities of moving from traditional expository teaching to classroom cultures where learners' ideas are valued and form the basis of teaching, with the result that learners become autonomous thinkers.

To a large extent, these cultures are recognised as frequently existing in Japanese elementary school classrooms (see, for example, Doig, Groves, & Fujii, 2011) and are being promoted in many countries through the adoption and adaptation of Japanese Lesson Study. In our current research project *Implementing Structured Problem-Solving Mathematics Lessons Through Lesson Study*[6], a major change in pedagogy in teachers' research lessons, and to some extent in their everyday practice, has been the incorporation of an extended, clearly orchestrated, whole-class discussion of students' representations and strategies. Arguably, these discussions provide opportunities for students to develop their conceptual understanding, strategic competence, and adaptive reasoning.

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[6] *Implementing structured problem-solving mathematics lessons through lesson study* is funded by Deakin University's *Centre for Research in Educational Futures and Innovation*. The project team is Susie Groves, Brian Doig, Colleen Vale and Wanty Widjaja.

It will be interesting to see the kind of impact frameworks such as those discussed in this paper will have on classroom practice in Australia and elsewhere.

Note: This paper originated as a keynote address at the *Persidangan LINUS Antarabangsa* 2012 [International Literacy and Numeracy Conference] held in Kota Kinabalu, Sabah, Malaysia from 26 - 28 November 2012.

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